The Common Core Mathematical Practice Standards in the DNA of *Think Math!*

The first edition of *Think Math!* was published four years before the final draft of the Common Core State Standards for Mathematics (CCSSM) was first released in June 2010. You might think we'd now have to play catch-up. In fact, that is not the case.

While some *content* details of the CCSSM could not possibly have been known before the standards were released, two principles central to the design of the Common Core Standards were already widely accepted. The first principle was that mathematical proficiency requires two things: fluent knowledge—the facts, vocabulary, procedures, formulas, and theorems that we associate with math courses—and the habits of mind that allow mathematicians (and others) to generate that knowledge. The second principle was that to build mathematical proficiency, courses needed greater focus. The many disconnected bits of the "mile-wide, inch-deep" curricula that had evolved over the years gave students neither the time to develop comfort and skill with key ideas and practices nor the coherence to make sense of those ideas and assemble them into a mental discipline.

These two concerns were part of the very DNA of *Think Math!*, driving principles behind its design from the time we first proposed it to the National Science Foundation in 1999. In fact, our concern about the mathematical thinking that the CCSSM now mandates in its Mathematical Practice standards had been central to our work at EDC even earlier. We published "Habits of mind, an organizing principle for mathematics curriculum" in 1996 after roughly four years developing and applying those ideas. The ideas we first articulated in 1996 and extended in many subsequent articles have found a home in the CCSSM Mathematical Practices.

Our concern for finding ways to bring focus and depth to curricula—despite the pre-CCSSM plethora of topics required by disparate state frameworks—was a consequence of our determination to build both the skills and the thinking of mathematics. *Think Math!* was designed around a highly focused set of curricular goals, but achieved those goals through a mathematically rich and diverse set of examples. Not only *could* mathematical contexts vary richly while emphasizing a tightly focused set of goals, they *had* to vary in order to develop depth of understanding and in order to make the necessary practice intellectually interesting.

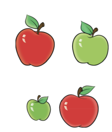
These two concerns also drove the thinking behind the CCSSM. Because mathematical proficiency is needed not only for science, technology, engineering, and mathematics but also for auto diagnosis and repair, medical diagnosis and treatment, social science research and business, architecture, construction, and the management of one's own money in an age of financial complexity—virtually all aspects of modern American life—curricula and instructional plans must have sufficient depth and focus to help students learn to *think mathematically* and also gain *skill and proficiency*. Accordingly, the CCSSM presents two kinds of standards: Standards for Mathematical Content and Standards for Mathematical Practice. The content standards are listed in a way that is, in its form, not unlike lists we've seen in state frameworks for years, but these new lists are more lean and focused, and represent "a balanced combination of procedure and understanding" with a mandate to connect the practices to the content. By contrast, the standards for Mathematical Practice, despite the long history of thought behind them, are the "new" element, less familiar to most people. They outline critical aspects of mathematical thinking: "…ways in which developing student practitioners of the discipline of mathematics increasingly ought to engage with the subject matter as they grow in mathematical maturity and expertise…" (CCSSM, 8).

Mathematical Practices

The eight Mathematical Practice standards complement the content standards by aiming curricula, teaching, and tests at helping students to see their world through a mathematical lens, reason mathematically, and *use* mathematics thoughtfully and effectively to solve problems. All eight are addressed throughout *Think Math!*, some on a nearly daily basis. In this paper, I will illustrate that with two of them.

1. **Make sense of problems and persevere in solving them.** The CCSSM describes this standard by saying that "mathematically proficient students … [look] for entry points… analyze givens, constraints, relationships, and goals. … They monitor and evaluate their progress and change course if necessary." A major part of problem-solving is figuring out how to begin. Real-life problems don't ask what chapter you're in; they just appear. And you must figure out what information you have, what you need, where to start, and what to do. Whether you are an engineer, a teacher, a doctor, an auto mechanic, or a mathematician, you always face new situations for which you must puzzle out where to begin and what action to take, and then watch to see how it changed the situation, and what might be sensible to do next.

   In *Think Math!*, this kind of sense-making and puzzling-through is part of students' everyday experience. Most of the Headline Stories ask students to analyze the givens and make sense of them, deriving new mathematically relevant statements or posing new questions about the situation. Kindergarteners discussing this picture will describe it as four apples, or three big apples and one little one, or two red and two green, beginning the process of learning that more than one mathematically relevant statement can often be made about a single situation. (More about Headline Stories at http://thinkmath.edc.org/index.php/Headline_stories.)

   The Cross Number Puzzle is another way in which *Think Math!* integrates content, mathematical representation, and reasoning. In this puzzle, the sum of the numbers on one side of the heavy line in any row or column must equal the sum of the numbers on the other side of that heavy line in the same row and column. The *content* is the logic of addition and subtraction and the role of place value in calculating using the standard algorithms. The *representation* is a table, a mini-spreadsheet. This table organizes kindergarteners' sorting, second graders' adding and subtracting, and fifth graders' division, and its logic prepares students for reasoning about systems of equations when they reach algebra. But the reason we present this tabular form *as a puzzle*, aside from pure appeal, is that the nature of any good puzzle is about "looking for entry points," figuring out how to start and what to do next. Thus, grade-appropriate elementary school content is presented in a way that reveals an underlying mathematical structure that prepares students for algebra, and in a form that evokes and teaches an essential problem-solving process. (For more about Cross Number Puzzles, see http://thinkmath.edc.org/index.php/Cross_number_puzzles.)

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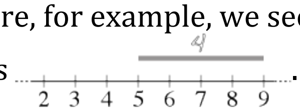
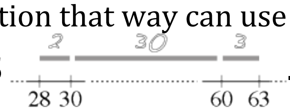
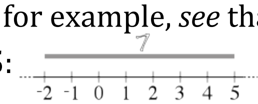
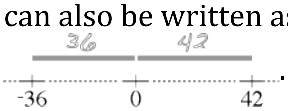
5. **Use appropriate tools strategically.** The CCSSM makes clear that these tools are not just physical tools such as "pencil and paper, concrete models, a ruler, a protractor, a calculator, a spreadsheet…" but also mental tools such as "estimation and other mathematical knowledge." Along with one of the most powerful mental tools of all—the puzzler's disposition of first assessing what one knows and thinking strategically about where to begin—*Think Math!* provides three particularly powerful mathematical tools that are appropriate in the earliest grades and remain valuable throughout high school mathematics and beyond. One, the early and continued emphasis on tables and, in particular, the spreadsheet-like table we call the Cross Number Puzzle, has already been mentioned.

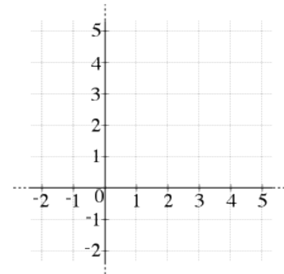
The second important tool, the number line, is sometimes regarded just as a visual aid for children; it is, in fact, a sophisticated image used even by mathematicians. For young children, it helps make early mental images of addition and subtraction that connect arithmetic with measurement. Rulers, another tool that students will get plenty of chance to use, are just number lines built to spec! Here, for example, we see "the distance from 5 to 9" or "how much greater 9 is than 5" as 

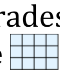
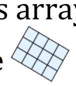
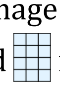
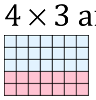
. Children who see subtraction that way can use this model to begin to see "the distance between 28 and 63" as 35 

. Many learn to see that in their heads, too, and do this subtraction mentally. This is essentially how clerks used to "count up" to make change. Because the number line model extends so naturally to decimals and fractions—just by "zooming in" to get a more detailed view of that line between the whole numbers—and extends equally naturally to negative numbers, it *unifies* arithmetic, making sense of what is otherwise often seen as a collection of independent and hard-to-remember rules. We can, for example, *see* that the distance from $^{-}2$ to 5 is the number we must add to $^{-}2$ to get 5: 

. And we can *see* why 42 – $^{-}36$ can also be written as 42 + 36: the "distance from $^{-}36$ to 42 is 

. The number line remains useful as students study data, graphing, and algebra: two number lines, at right angles to each other, label the addresses of points on the coordinate plane.



The area model of multiplication is another powerful tool that lasts from early grades through college mathematics. For second graders in *Think Math!*, images like  along with questions like "how many columns, how many rows, how many little squares" help establish the small multiplication facts. So might pure drill, of course, but this array image goes much farther. Seeing the same array held in different positions like  and  makes clear that we can label any of these 3×4 or 4×3 and the number of little squares is always 12. In grades 3-5, array pictures like  help clarify the distributive property of multiplication, the property that makes multi-digit multiplication possible, makes sense of the standard multiplication and division algorithms, and underlies the multiplication that students will encounter in algebra. In this picture, we see that “two 7s plus three 7s is five 7s” or $2\times7 + 3\times7 = (2 + 3) \times 7$. A schematic version of this image organizes students’ thinking as they learn multi-digit multiplication; this image, combined with the Cross Number Puzzle, models the conventional algorithm exactly, making total sense of what can otherwise feel like an arbitrary set of steps. The same image also allows students to acquire and understand the algorithm for division as a process of “undoing” multiplication, greatly simplifying the learning of a part of arithmetic that has a long history of being difficult. What makes this a *powerful* tool is that it serves the immediate goals of elementary school arithmetic in a way that prepares students for algebra. Algebraic multiplication, which has no “carry” step, is modeled perfectly by exactly the same tool. (To learn more about how *Think Math!* develops the use of the area model tool, see http://thinkmath.edc.org/index.php/Multiplication and http://thinkmath.edc.org/index.php/Multiplication_and_division.)

This versatile suite of tools builds *mental* models that last. What makes a tool like the number line, area model, and table/spreadsheet truly powerful is that it is not just a special-purpose trick or temporary crutch, but is faithful to the mathematics and is extensible and applicable to many domains. These tools help students make sense of the mathematics; that’s *why* they last. And that is also why the CCSSM mandates all three of them.

Of course, *Think Math!* also helps students make strategic use of *physical* tools. When students first study decimals, they do it with a “native speaker of decimal,” the calculator. The calculator does not *replace* the students’ thinking; as a native speaker, it helps them learn the new language, the decimal notation.

And students learn paper and pencil methods, as well. Because the standard algorithms are efficiently honed, compact notations, they hide the logic students need to learn for algebra. While that makes them poor ways of *acquiring* the logic of computation, that compactness is also their great virtue *after* one understands the logic. By reducing the “cognitive load”—the amount of thinking we need to do in order to perform a calculation—they raise speed and accuracy for calculations that we cannot quite keep entirely in our heads and that we don’t do with a calculator. So these are taught, but not as “how to calculate”; instead, they are taught as a *summary*, a compact notation for the processes students have already acquired logically. That way, students may choose between mental methods, various uses of paper and pencil, and calculator, rather than defaulting to the calculator as the only easy way or to paper and pencil algorithms as the only “legitimate” way.

Content standards: skill and understanding (teaching focused content through mathematically rich and diverse contexts)

> The Standards for Mathematical Content are a balanced combination of procedure and understanding. (CCSSM, 8)

Popular debate has often pitted skill and understanding against one another, as if time spent on one steals time from the other. But this is not the case. No tradeoff is required and, in fact, it is not generally *possible* to have one without the other.

There are, of course, facts and skills that you acquire without "understanding." Your friends' names are such facts, and walking is such a skill; both get constant enough use that nothing else is needed to sustain them. Salience—from constant use, from a sense that it is important, or just because it interests you—is what makes a fact stick. For fluid skill—in mathematics, violin, or skating—practice is also essential. Some students find the details of, say, long division sufficiently salient that they can acquire and retain that skill without knowing why it works. But many students cannot and, even for those who can, the skills are difficult to maintain without conceptual scaffolding to prop them up. Rules that seem arbitrary are very hard to remember correctly.

It is equally possible to develop certain kinds of understanding without skill—for example, you can understand roughly the way a car works without knowing any of the fine details or having any skill at fixing it—but mathematical understanding is hard to develop, quick to fade, and nearly impossible to apply without skill. You can't recognize a numerical pattern if you don't already have in your head the fluent knowledge that allows you to see how the numbers are related; you can't see the underlying sense of a mathematical procedure if all of your attention is taken up just with managing the details. And, besides, the very purpose of "understanding mathematics" is to help you *do* things with it.

How do you build these two at the same time? Both *Think Math!* and the CCSSM, at different times, did it the same way. The CCSSM chose content strategically and calls for connecting that content with mathematical practices. The *Think Math!* designers aimed for almost exactly the same content, with very few exceptions. And we succeeded in focusing on those topics, despite having to meet state standards that still mandated a wider spread, by delivering some of the topics we considered less central *through* practice or application of the central ones. By organizing the curriculum around big mathematical ideas—not isolated topics, but pervasive themes, central properties, mathematical habits of mind, and what the CCSSM calls Mathematical Practices—"distinct" topics could support each other rather than compete with each other for time. For example, area is listed in the CCSSM under "Measurement," but the CCSSM explicitly calls for it to be tied to ideas about multiplication and the distributive property. That was exactly the way *Think Math!* designed both topics to connect.

Handling one topic in a way that serves another is also how *Think Math!* can stay completely focused on elementary school mathematics—focused and *deep*—and yet do it in a way that prepares students well for the algebra that they will learn in later grades. Even the Mental Math (Skill Practice and Review) exercises that we provide—essentially drills—are strategically designed for that purpose. The more familiar kind of random fact drill depends entirely on rote memory and builds nothing more; *Think Math!*'s mental math exercises build an intuitive sense of all of the central properties of arithmetic. Doubling and halving everything practices mental use of the distributive property; multiplying by 5, by multiplying by 10 and then halving, practices the associative property of multiplication; practicing pairs to 10 and to 100 and (later, with decimals and fractions) to 1 develops key ideas about complements and the ability to maintain a mental number line. Students who get this kind of practice develop astonishing facility with mental computation, but they get way more than that: they build "gut" familiarity with all of the properties of arithmetic that they will need for algebra.

Handling content while maintaining a clear focus on mathematical ideas and practices became a particular challenge as we were designing ways to meet the prior frameworks' calls for a vast set of seemingly arbitrary geometric vocabulary. We could just teach it, of course—children can memorize those words as easily as they can memorize their classmates' names—but teaching disconnected terms felt anti-mathematical and, besides, took time away from content we (and later CCSSM) considered core. So we took on the fascinating challenge of inventing a mathematically significant activity—something that served more central goals of *Think Math!* and was interesting to children—that would naturally *need* those terms for clear communication. (Using words in communicative context is, anyway, how children acquire vocabulary best and how they've acquired the vast bulk of their vocabulary which, by first grade, is already half what their adult vocabulary is likely to be.) In our 3-D geometry chapters, each student in the class folds a unique net, making a very diverse set of 3-D shapes, as many different ones as there are children in the class. Because the set *is* so diverse, the individual objects are not trivial to name; describing your own creation requires careful analysis of its features. Naming the number and shape of the faces practices 2-D shape vocabulary. Making other observations and inventing puzzles about the shapes requires new 3-D vocabulary. Counting vertices, edges, and faces is no longer an arbitrary requirement, but part of a well-motivated exploration of the nature of prisms and pyramids, and other polyhedral objects that are neither of those, making some intriguing discoveries, and communicating all of that clearly. The CCSSM (Mathematical Practice 6) says "Mathematically proficient students try to communicate precisely to others." Our activity guarantees not only that students *have* usefully precise descriptive ideas and vocabulary, but also have the motivation to be eager to communicate that to others—as the CCSSM puts it, to "*try* to communicate precisely." The fact that each shape is unique, and that each child "owns" a shape, puts a great premium on being able to *describe* your own shape with precision. We also try to make clear to the teacher, as well as to the student, what is (and even what isn't) essential about the vocabulary. (To see how *Think Math!* treats some mathematical terms, see http://thinkmath.edc.org/index.php/Face, http://thinkmath.edc.org/index.php/Width, and http://thinkmath.edc.org/index.php/Right_angle.)

These references, in order by date, are a selection from our work on mathematical habits of mind and the design of curriculum

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