Mathematical Habits of Mind for Young Children

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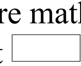
Elementary school teachers are in the unique and difficult position of managing just about everything, and often try to integrate their activities, finding ways to draw math lessons from literature, science, art, or lunch-money collection, finding ways to draw language lessons from science or math contexts, and so on. “Wouldn’t it be nice,” goes the theory, “if what we do at reading time, or in the morning calendar activity, or in our study of the chicks helps the children use and practice what they are learning in math? And wouldn’t it be nice if the math lessons gave the children new and useful vocabulary, and practice with their reading and writing?”

An advantage of this way of thinking is that it leads us beyond the particulars of each subject and helps us consider what children really need, and keep that in mind in all the contexts that we have—mathematical, scientific, linguistic, artistic, and so on. For example, it’s pretty basic that people need to be able to communicate clearly. This includes not only feelings and the plot lines of stories, but also procedures and instructions, and logical, relational, quantitative, and spatial information, too. Thus, it would be hardly a criticism to say that much of the typical elementary school mathematics *content* is no more mathematical than it is a specialized area of the language arts curriculum! Knowing that  is called “rectangle” is no more mathematical than knowing that  is called “boat.” The many areas of mathematically-oriented communication— quantitative, logical, relational, spatial, and so on—involve a fair amount of vocabulary, and considerable care in the use of words. When we teach this, are we teaching mathematics or language? It really doesn’t matter what we call the subject. The kids need it, so we teach it.

But even though some of what we teach in mathematics is appropriately not unique to that subject (or especially mathematical), we must avoid one subtle risk of “integration”: that what truly *is* unique about mathematics can get lost altogether. This chapter focuses not only on how mathematics reflects important ways of thinking that we’d like *all* subjects to support and *all* children to acquire, but also on how mathematics especially hones, refines, and extends these ways of thinking. This dual focus attempts to answer why we should, and how we can, teach *mathematics* that truly serves *all* students. The *why* is easy enough: A “mathematics for all” curriculum must include the facts and basic skills that everyone uses, and it must teach them well enough so that people are mathematically enabled, not disabled. It must support and refine the kinds of reasoning that *all* people need, regardless of their eventual choice of career or life style. And it must lay a foundation that allows anyone to *choose* pursuits that require more advanced

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mathematical ideas than the vast majority of us will ever use[1]. Without these ingredients, the curriculum either does not really involve mathematics, or it is not for all. *How* to teach a "mathematics for all" becomes clearer when we recognize that mathematics is both *a body of facts* accumulated over the millennia, and *a body of ways of thinking* that has allowed people to discover or invent these facts and ideas. In our view, teaching these ways of thinking, which we refer to as *mathematical habits of mind*, is a vital part of mathematical instruction at every level.

In this chapter, we discuss some of the habits of mind that are especially relevant in grades K-5, and offer some concrete suggestions on how they fit into everyday teaching and learning.

Some Habits of Mind for the Elementary Grades

The sections that follow describe five habits of mind, their roles outside of mathematics, and how mathematics extends, specializes, or sharpens each of them. Because the language arts curriculum is such an important part of elementary schooling, and because communication is such a good example of a broadly important skill that math can help develop, we focus a lot of attention on communication-related habits of mind. Our hope is to help you step beyond the particular examples we give, so that you can apply the principles throughout your teaching.

Habit of Mind 1: Thinking about Word Meanings

Creating, testing, adjusting, and working with definitions is a major part of mathematics. Though children rarely need to create formal definitions, all of the skills of doing so are part of expressing oneself clearly. A definition—especially outside mathematics—is hard to do right, as it must include all cases of the thing to be defined but not allow anything extra. Consider, for example, trying to define *chair*. Little children might just give it a utility: "You can sit on it." But pillows, tables, horses, and the floor also fit that description. A category, such as "a piece of furniture," also doesn't do, by itself. It may help to combine the two, or to say not just that you *can* sit on it but that "its *primary purpose* is for sitting on." Trying to *make* definitions helps children (and the rest of us!) see why they can be so difficult to *understand*. How often have you seen a child look up a word in the dictionary, perhaps even copy the definition faithfully, and then write something like: "At night, we extinguished [put out] the cat."

It is valuable for children to become attuned to the ways words are used, whether for the logical reasoning that mathematics requires or just for the sake of expressing themselves clearly. Mathematics gives many opportunities for thinking about and playing

[1] This is a tough distinction. Mathematical ways of thinking—even some of the most sophisticated ones—are often used by people who don't use "mathematics" at all in their work or pleasure, and who consider themselves poor at and ignorant of mathematics. On the other hand, even such a mathematical big deal as the Pythagorean theorem—which is a big deal precisely because it pops up all over the place in mathematics (not just in geometry), in forms that sometimes disguise it quite well—is not something that most of us *ever* use explicitly unless our business is the doing or teaching of math, science, or engineering.

with the way words are used. Think about the meanings of the words *circle* and *on* in these two contexts:

- Cut out a circle. Place one dot anywhere on that circle.
- Draw a circle. Place one dot in the circle, and one dot on it.

A definition creates a category, and categories may be related. Many mathematics curricula and high-stakes tests expect elementary school children to know the hierarchy of categories *polygon* $\supset$ *quadrilateral* $\supset$ *parallelogram* $\supset$ *rectangle* $\supset$ *square*[2], as in this problem adapted from a teacher-made fourth grade assessment test.

4 Look at these pictures.

List all polygons. *all but E*

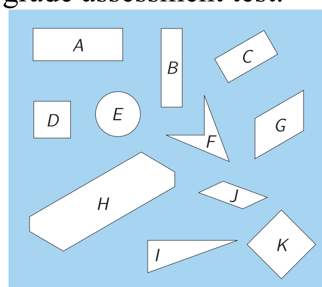
List all triangles. *only I*

List all quadrilaterals. *all but E, H, and I*

List all parallelograms. *all but E, H, I, and F*

List all rectangles. *A, B, C, D, and K*

List all squares. *D and K*



Three of these words are frankly of little use to most people outside of school, but there are two worthy ideas that one can learn from exercises like this. One is that *there are hierarchies of categories*. This is hard for kids, both in and out of math. Faced with six toy cows and four toy horses, young children regularly trip over questions like "Which is more, the number of cows or the number of animals?" The problem is not with counting. Asked independently how many animals there are, they'll get the right answer. For very young kids, calling a set of creatures "cows" seems, temporarily, to exclude those creatures from other categories. Does this mean that young children need specific lessons on quadrilaterals to rectify this problem? No! They'll eventually begin to understand multiple levels of classification anyway. But experience at a developmentally appropriate level helps.

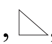
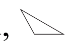
A second useful idea is that *the way categories are used depends on the context*. When we ask students to mark all the rectangles on an exam, we expect them to include the squares. But when we ask students to *draw* a rectangle, we may well think they've misunderstood, or have even been a bit obstreperous, if they draw a square! It is not the category "rectangle" that has changed from the exam context to the drawing context, but the way we use it. The exam item checks whether students understand that the *general* category "rectangle" includes the *special case* "square." The request to draw a rectangle—whether in math class or in casual conversation—calls for a picture that

[2] The $\supset$ sign means "includes." Like the > sign, it opens up to the "larger," more inclusive category. So, things that we'd call polygons (roughly speaking, closed shapes that have only straight sides) include all the things we'd call quadrilaterals (four-sided polygons), and so on. At this point, *English* begins to be a bit confusing. It is correct, in English, to say "Squares are rectangles," but "are" clearly does not mean "=." A comparable statement outside of mathematics is "Bears are animals": *animals* $\supset$ *bears*.

suggests the *general* category, not the special case. Neither view is superior; the context determines what is appropriate.

Children need to develop their sensitivity to contexts and to the kind of generality, inclusivity, and specificity appropriate to each. Their non-mathematical writing often suffers because they use words like "thing" or "went" that are not specific enough to make their writing interesting, or, at times, even understandable. (At other times, they err on the side of over-specifying something that is already obvious to the reader, as in "a triangle with three sides.") The study of mathematics is not the only way to appreciate the significance of context. Other subjects (especially, and obviously, language arts) are equally good or better venues. But contexts and categories in mathematics are not as "messy" as in the "real world," making it easier for children to play with them. As long as tests (and many curricula) place so much emphasis on vocabulary—which, from a purely mathematical point of view, is an unwarranted and misleading emphasis—we can seize the opportunity to teach good thinking.

Another important idea about categories is that they should be both *useful* and *testable*.

Usefulness. It can make sense to treat *different* objects as if they were the same in a context in which, for some practical purpose, they *are* the same. If enough such contexts exist, we define and name a category. These shapes— , , , , and  — are all different, and yet they have *so* much in common, so much can be said about *all* of them, that they are worth treating as a group, even (for some purposes) as if they were all the same thing. For example:

- The area of each figure can be computed by multiplying the length of its horizontal side by the height of the figure and then dividing by 2.
- The angle sum in all of these figures is the same.
- If you draw any other closed figure with the same length sides as one of these, the new figure will exactly match the old one, angle for angle.

The category *triangles* is *useful*, and so it warrants a name. But, except for passing tests, a category *doesn't* warrant a name until one has something worth saying about it. When the only thing your students can say about a rhombus or quadrilateral is whether some figure *is* one or not, they get no more than a test answer out of learning the term.[3]

How inclusive or exclusive should a definition be? For example, why is a square *in*cluded in the family of rectangles? And why is 1 *ex*cluded from the family of primes?

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[3] On the other hand, the word "quadrilateral" can be a great exercise in spelling and etymology for young children—a good use for the lesson if one must teach the term anyway. The origin of words *does* interest many elementary school children. Drawing connections between the technical vocabulary of mathematics and everyday (or, at least, non-mathematical) vocabulary can help *both* the learning of the specialized mathematical terms and also *general* vocabulary building purposes. Quadrilateral, for example, is built from *quadri* (four) and *lateral* (side). Number-word-parts like *quadri* show up in many English words: quadrant, quadrangle, quadruped, quadruple, and even (slightly more disguised) in quarter, quart, square, and dozens of other words. "Fancy" vocabulary also tends to draw heavily from Latin and Greek, providing wonderful opportunities for students who know or are learning languages that derive from these to contribute their special knowledge. For example, in Spanish or Portuguese, four is *cuatro*, and in French it is *quatre*.

The reason why definitions are built to be neither too exclusive nor too inclusive is to make the categories *useful*.

Rectangles are often *described* as if the sides must differ, but they should not be *defined* that way. Saying that sides "don't have to be equal" is not the same as saying that they "must differ."[4] Except for an *added restriction* (sides must be equal), the square shares all the properties of a rectangle, and everything that can be said about a rectangle (like area formulas) applies equally well to a square. Therefore, it is *useful* to define rectangles in a way that includes squares.

In a similar way, 1 is *defined* not to be prime because the notion "prime" becomes less useful if it includes 1. For example, non-prime whole numbers can be expressed as the product of a *unique* set of primes (equivalently, "factor trees" don't go on forever). We could not say that if 1 were considered prime: the prime factors of 12 would still be $\{2, 2, 3\}$ (because $2\times2\times3=12$), but also $\{1, 2, 2, 3\}$ and $\{1, 1, 2, 2, 3\}$ and so on. This makes the whole idea of "prime" less useful.

Testability. A good definition should make it easy to test whether a particular object fits the definition. Sometimes, thinking about the test helps make the category clearer. For an example outside of mathematics, consider this: A noun can be *defined* as "a person, place, or thing (or idea)," but a good test to see whether some word is a noun might be to see whether we can put *the* or *our* before the word, or *is* after the word. If we can say "the *x*" or "*x* is," then *x* is acting like a noun. A verb is often described as "an action word" but *is* or *thought* or *can* or even *sat* hardly strike children as involving much action. We can test to see whether some word is a verb trying to put *she* or *they* before it. If we can say "she *x*" or "they *x*," then *x* is acting like a verb. Similarly, if we can say "the *x* cat," *x* is acting like an adjective.

It is good practice in clarity of thought and expression for students to invent definitions that are spare, capture all correct instances, exclude all others, and are testable. In mathematics, this also helps students understand the concept more deeply than when they merely *use* definitions.

Do all categories have independent definitions? No. Look up any simple word you like (like "chair") in a dictionary. Then look up all the significant words (like "furniture") used in its definition. Continue in this way. Pretty soon you'll hit a loop.

To keep mathematical statements unambiguous, we restrict ourselves to use only words that are already defined or ones that we just plain agree we all know. Why is it impossible to use only *defined* words? When asked this question, a student would probably be able to provide a convincing proof, maybe by contradiction. Perhaps it would be something like: What would the definition of the first word consist of?

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[4] This is an important idea; algebra beginners often misinterpret $a + b$ as necessarily meaning a sum of *different* numbers.

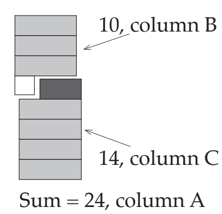
Habit of Mind 2: Justifying Claims and Proving Conjectures

Without some version of proof, much of what children learn remains unexamined and unexplained, a set of memorized facts and patterns, little black boxes that keep math a mystery. But this needn't be the case. Age-appropriate proof can—we believe *should*—exist throughout the elementary curriculum. Second graders can not only notice (conjecture) that the sum of any two odd numbers seems to be even, but can even come up with excellent explanations—true proofs despite the lack of formal trappings—of why this *must* be so, even for pairs of odd numbers they haven't tried. Some say (in second-grader style) that even numbers are just a bunch of pairs, whereas an odd number is a bunch of pairs and one left over. Adding two odd numbers gives a bunch of pairs and two left over, which make a pair, so it's really "just a bunch of pairs." Others may try to show a picture of that, perhaps using concrete materials.

When children begin to think about skip counting by three, they are ready for a fancier experiment, in which investigation, practice, and logical reasoning all come together. Consider doing the following activity.

Column A	Column B	Column C
0	1	2
3	4	5
6	7	8
9	10	11
12	13	14
15	16	17
18	19	20
21	22	23
24	25	26
27	28	29
30	31	32

1. Write all the whole numbers through 32 in three columns, as shown. What's the same about all the numbers in (any) one column? Pick any two numbers in column *A* and add them. Which column contains the answer? Try that for a different pair from column *A*. What pattern do you see in your results? How might you explain that pattern?
2. Do the same experiment on column *B*. Pick two numbers in column *B* and add them. Which column contains the answer? Why is it always that column?
3. Where do you find the answer if you add a number from column B to one from column *C*?



Long before students have formal algebra to answer these questions, they can use *equally generalizable logic,* with Cuisenaire® rods as a kind of "notation." They might, for example, show that all numbers in column A can be constructed using only light green rods (length, three units); that numbers in column *B* require one additional white rod (one unit); and that numbers in *C* require a red rod (two units). The answer to problem 3, then, could be given for any *particular* example with a construction or picture like the one shown here for 10+14 = 24, but the students can see that any other example would work the same way.

Would students "naturally" do this without help or prior experience? Very few, but most students *can.* Giving them such experiences in class helps them develop the habit of looking for patterns, the skills to find them, and the expectation that patterns have explanations. In a similar way, even knowing only what a fourth-grader knows, many students can explain (if asked, and if they've had chances to think about questions like this before) why a product like 109872388 × 23598109 must be even. Questions like this *should* be asked, as part of a strategy for building the expectation that conjectures and their proofs are part of the math game.

One group of third graders we know love a set of logic puzzles about two families: the Liars (who can never tell the truth, not even by accident!) and the Truthtellers (who can't lie, even in error).[5] For example, "You meet Dale and Dana. Dale says 'We're both Liars.' What family is Dale from? What about Dana?" Playing with puzzles like this gives students experience with looking for contradictions (could Dale be telling the truth?), tracking logical consequences, and (sometimes) trying all the cases (being systematic enough to know what all the cases *are*). Many students find it positively funny to consider what to make out of the statement "I am a Liar."

Here's another example. Fifth grader Naomi proudly proclaims that she has discovered a new math rule: that whenever the perimeter of a rectangle increases, its area also increases. She uses a table to demonstrate her rule:

	Rectangle 1	Rectangle 2	Rectangle 3	Rectangle 4
length	4 units	6 units	8 units	10 units
width	2 units	3 units	4 units	5 units
perimeter	12 units	18 units	24 units	30 units
area	8 square units	18 square units	32 square units	50 square units

Her rule seems to make sense; it looks like it might be true. Students might start by trying to find other rectangles to show that the rule is true. They are trying to prove her rule by showing lots of examples that support her claim. But, if they are systematic and planful in their choice of examples (or even by mistake), they are likely to find rectangles that contradict Naomi's rule. A *proof by counterexample*, a truly powerful kind of justification! This need not even be a defeat for Naomi. It is harder, but possible, to ask *under what conditions* her rule *is* true! This might lead students to think about shape, and is a good foreshadowing of ideas of geometric similarity.

As long as the patterns are interesting enough, elementary school kids love finding them and explaining why things are the way they are. And, when the proofs and proof styles are developmentally appropriate, the kids can often learn to do them very well. The curiosity to know what *and why*, and the skill to find out, are important to foster and develop in kids, not just as a precursor to proof in high school geometry, but because these traits are valuable outside of mathematics as well as within. Whether one becomes an investigative journalist, a mechanic (auto diagnostician), a doctor (human diagnostician), or a scientist, these same inclinations and skills are essential. Kids of 9 and 10 playing four-square at recess may spend more of their time debating the rules, or whether a play met the current rules or not, than in the actual play. They positively thrive on *looking for and assessing criteria* and on *proving things right or wrong*.

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[5] Adam Case's *Who Tells the Truth?* has a nice set of these Liar/Truthteller puzzles. This kind of puzzle, famous in mathematics, is found in many recreational math books.

Proof within mathematics depends entirely on logic applied to previously agreed truths, whereas proof outside of mathematics often accepts data (evidence), likelihood, and precedents, as factors. Even so, the fact that the same word is chosen for both styles of thinking suggests that these two kinds of "proof" have something in common. Mathematics should not *replace* the kind of logic kids otherwise use, but should surely *augment* it, especially starting at a time when kids naturally "study" this kind of argumentation on their own! As students learn mathematical proof, they should consistently see how it relies on—and builds on and refines—their own good sense.

Students don't need to know the names of different kinds of proofs, or any such formalities. They do need to recognize the difference between a wild guess, a conjecture, and a proven assertion. And they do need to develop the inclination to wonder *why* things are as they are, to *expect* reasons, and—when possible—even provide a logical chain of reasons as the explanation. They also need to experience a variety of proof styles (expressed in developmentally appropriate ways). These might include any of the styles shown above: the constructive build-a-model approach with the Cuisenaire rods to investigate the skip-counting-by-three experiment; the raw logic (or exhaustive testing) that works on the Liar-Truthteller puzzles; and the healthy skepticism that leads students to find a counterexample in the perimeter and area problem.

Habit of Mind 3: Distinguishing between Agreement and Logical Necessity

Life outside mathematics involves rules, for example, the rule of not interrupting a person when that person is talking. This rule is purely a social convention, a "given." It can even be broken without causing any trouble at all when friends are close and conversation is casual—that is, when all parties agree. The source of authority is just *social agreement*.

Mathematics also involves rules. For example, in late elementary school, many curricula teach what is called "the order of operations." In an expression like $3 + 4 \times 5 + 1$, one might imagine performing the arithmetic from left to right, adding 3 and 4, then multiplying by 5, and finally adding 1 to get 36 as a result. By *agreement*, however, we don't do that: we perform multiplication and division before addition or subtraction. In this case, that means that the 4×5 is done first leaving us with $3 + 20 + 1$, or 24, as a result. Why *this* rule is chosen is pure convention, designed, like definitions, to serve only one master: usefulness. The source of authority is *social agreement*, for the sake of clear communication and useful results.

Now consider another mathematical rule, the rule that two lengths may be added only if they are expressed in the same units. Three feet plus 24 inches is not 27 of anything: it is either 5 feet or 60 inches (or 1.666… yards, and so on). Likewise, three nickels and two dimes cannot be added until the units are the same: they can be jointly regarded as coins, in which case the sum is "5 coins" or they can jointly be regarded as cents, in which case the sum is "35 cents." This rule is not based on social agreement; it is *logically necessary* in order for addition to make sense, and (therefore) it applies *across the board* to addition.

So, fractions must follow the rule: halves and thirds cannot be added until they are converted to the same units. One such common unit is sixths: three sixths and two sixths are five sixths. The same goes for the standard algorithm for adding two multi-digit numbers. The reason to line up the columns is not because "that's the way it's done" but because that helps make sure one only adds things that are expressed in the same units. The reason this "sum of 120 and 43" is wrong is that it tries to add a 2 and a 3 (and a 1 and 4) that represent different things. The 2 could be tens, or dimes, and the 3 is ones, or pennies. The source of authority for all these addition rules is *logical necessity*. You can't add apples and oranges!

$$\begin{array}{r} 120 \\ +43 \\ \hline 550 \end{array} \text{ oops!}$$

Why is it so important to recognize the source of authority—convention or logic—for each mathematical idea? Because one goal of school mathematics is to support the development of children's reasoning. If all rules are arbitrary, or if no distinction is made, rules become divorced from, or even the enemy of, common sense. How often do we see people use (or cave in to) data and graphs, even when the "math" does not really support the argument!

The fact that three-sided polygons are called triangles (among English speaking people) is also *social agreement*. So is notation, like the convention that the vertices of a triangle are labeled with capital letters, like A, B, and C, or that a side of ΔABC is written as $\overline{AB}$, not $\overline{ab}$ or AB. The source of authority for 360° around a point (90° in a right angle) is that "that's the way we define the words *degree* and *angle* and *right* angle," and so on—again *social agreement*.

But the fact that the three angles within a triangle have the same total measure as two right angles (what we call 180°) is *not* a matter of social agreement, or even a "natural law" discoverable by experimentation. It is *logically necessary*: one can show (prove) that the angles *must* add up that way. Most often, you are shown this necessity by a proof in a geometry text, but if you know the conventions (and have a bit of experience doing the math), you can *reason* out this result yourself.

Conventions, like people's names and addresses and phone numbers, may have patterns and reasons and history, but not logical necessity. Mathematics provides a beautiful counterpoint, an opportunity to see that *some* truths—such as the fact that the sum of two odd numbers *must* be even—are ones that students can reason out for themselves. Mathematics gives them a chance to practice that kind of reasoning. And only in mathematics is this appeal to logic-alone possible.

Students commonly ask, in any subject: "Did I do it right?" "Is my answer correct?" Mathematics often lets students answer such questions themselves, helping them build a certain independence of learning and a confidence that can have benefits far beyond the math lesson. The student can say, "I can figure this out by myself, I don't need books or authority figures. *I* can be the authority." Seeing when this *is* true also help students learn to recognize when it is *not* true. Convention, cultural legacy, data from history or the physical world, and so on, are different, and require the authority of reliable outside sources (books, people, experiments).

Habit of Mind 4: Analyzing Answers, Problems, and Methods

Checking and analyzing answers. When we were kids, checking meant only doing the problem again, or working backward from the answer—checking a subtraction with an addition, for example. Checking has another, richer meaning—reviewing an answer for reasonableness. You can tell when students are trying to make sense when you hear chains of thought like these:

- I added two numbers, but my answer is less than one of them, so something must be wrong.
- My answer is three-and-a-third busses. That makes no sense!
- The area of rectangle A is 24 square inches, and I got 50 square inches for rectangle B's area. This looks OK because B looks much bigger. Oh, and if I place A here on top of B, there is about enough space for another copy of A next to it, so B is about twice as big as A. My answer seems correct.

Students must see that even where computations are required, common sense remains central.

Tinkering with the problem. When a student has found, justified, and made sense of an answer, it's time to step back and see how the whole problem (statement, solution, and answer) fits in a bigger picture. Can other problems be solved with the same method? How can the answer to this problem be applied elsewhere? Can this problem be extended or generalized to address new situations? This, in turn, leads to one of the most powerful ideas of all: that even (in fact, especially!) in mathematics, one can ask what-if questions. The what-if has to be taken seriously to see what its consequences really are. But only as children see that this is ok—that asking what-if questions is a genuine part of the "mathematical game"—can they feel completely free to suggest such questions themselves (see Goldenberg & Walter, 2002; Brown & Walter, 1990).

Creating and analyzing algorithms. There is, today, intense debate about whether children should be learning standard algorithms or making up their own. We're inclined to think that the reason the debate rages on is that there's wisdom on both sides: neither answer is sufficient by itself. On the one hand, a standard algorithm[6] gives students a general method for solving certain kinds of problems. On the other hand, the process of inventing approaches of their own involves students in analyzing the problems and answers and in developing the kinds of ideas—generally quite algebraic (though without the specialized notation)—that may help them understand and appreciate a general algorithm. In fact, creating a good algorithm requires reflecting on the steps required for solving a problem, then generalizing these steps in a way that can solve a class of similar problems. Each approach contributes something important.

These reasons for standard and student-generated algorithms are given from the teacher's perspective. There is also something we want students to realize as they work

[6] "*The*" standard algorithm is quite another story. Algorithms vary from society to society, and some are more efficient than the ones we call standard. For example, the addition algorithm we typically call standard in the US works from right to left, and is safe and efficient on large columns of multidigit numbers. If only two or three multidigit numbers are to be summed (the much more usual situation facing people now) a left-to-right approach is equally efficient, makes mental computation easier, and ties in more closely with other mathematical ideas and curricular goals such as estimation and rounding.

with algorithms: even routine computations can be done in more than one way, and the method for doing so may come from the student as well as from the teacher or the book.

Habit of Mind 5: Seeking and Using Heuristics to Solve Problems

Students need ways of approaching problems, even when the problems are unfamiliar and the students have no prescribed method for solving them. Techniques like Try-some-cases, Think-of-a-similar-problem, and Look-for-a-pattern, can give insight and move one closer to a solution but are not guaranteed methods (algorithms) for solving the problem. Such techniques are often called *heuristics*. Learning some heuristics and using them can build confidence, which makes students more willing to engage in problem solving, which gives them practice and makes them better problem solvers, thus leading them to seek and acquire yet more heuristics!

We won't add to the vast amount that has been written about the value of having a repertoire of mathematical problem-solving techniques that can be used flexibly and appropriately. Some of our favorite books on this subject—like Polya (1945) and Brown and Walter (1990, 1993)—have wonderfully useful lists. Instead, we want to make one point about the learning of heuristics. *You*—because you are an adult with long experience solving problems inside or outside of math, and also because you are engaged not only in solving problems but in teaching others how to solve them—*you* might benefit from focusing specifically on and learning a collection of heuristics. But for your students, that makes less sense. If they are not *doing* the problem solving, knowing heuristics won't help them; if they are, they will develop heuristics by applying them in context. Simply put, the way to learn problem solving is by solving problems.

This does not mean that you have no role in teaching the students how to solve problems. Your first contribution is to give students frequent opportunities to solve nonstandard problems of adequate difficulty.[7] Even if you do nothing else, this gives students a successful experience with problem solving. Your next contribution is to provide occasional hints to help students who are stuck. Sometimes a single remark such as "Would a table help?" "Try a simpler case," "How did you solve the other problem?" not only helps students solve the problem, but also solidifies their use of a particular technique so that they're more ready to use it in solving other problems. By introducing a heuristic just when they are stuck, you optimize its value to them.

One student's write-up on a problem said: "Once discovering this pattern, we were first at a loss as of what to do with this information. Just as we were in the depths of despair, the beautiful angel of mathematics [their teacher] came and said unto us, 'Try a chart!' We were inspired by this new angle on such a problem and quickly followed her suggestion. . . ." Most probably these students will remember the technique of making a chart forever.

[7] Nonstandard in the sense that the students do not already know a routine method for solving it. Of course, "adequate difficulty" means challenging enough to be a real problem, but still possible for them to solve with effort.

Taking the Habits of Mind Perspective as You Teach

Our purpose in focusing on what we call "habits of mind" is to strengthen the connection between plain common sense and the teaching of mathematics. We believe that a similar focus in the classroom has a similar benefit for students. By making mathematics support, extend, and refine common sense—not replace it or even supplement it with a set of clever but "magical" tricks and methods—helps to develop students' critical thinking, and their skills both inside mathematics and outside it. Just as we think it is important for students to expect there to be reasons behind mathematical patterns and facts, we think it is important for *you* to expect reason and sense behind claims made about mathematics teaching. We hope, therefore, that we've provided sufficiently reasonable support for the claims we've made about what is "important" for students and for teaching. *Thinking*, we think, must be at the core of all school learning—that is perhaps so basic to us it is like a postulate, something accepted *without* reasons—and the rest follows from that. Mathematics is certainly a discipline whose principal component is thinking.

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